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# Automorphism Groups of Orthomodular Lattices Obtained from Quadratic Spaces

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We study the automorphism group of some orthomodular lattices, obtained from a quadratic space over a field $K$. We show how this group is linked to the semi-orthogonal group and with the group of all similarity transformations of the quadratic space. When the field $K$ is finite, the cardinality of the automorphism group is given.

**KEY WORDS:** orthomodular lattices; quadratic form; automorphism group.

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## 1. INTRODUCTION

In Carrega *et al.* (2004), the authors introduce the orthomodular lattice (abbr. OML), denoted $T(E, \varphi)$, where $\varphi : E \times E \to K$ is a regular, symmetric bilinear form and $E$ a three-dimensional vector space on any field $K$. A structure theorem concerning the subalgebras of $T(E, \varphi)$ allows them to obtain infinitely many lattices $T(E, \varphi)$ which are minimal, that is, $T(E, \varphi)$ is not modular and all its proper subalgebras are modular. In Carrega and Greechie, the authors study the lattices $T(E, \varphi)$ when $K$ is a finite field. Here we study the automorphism group of the lattices $T(E, \varphi)$.

The general reference for Orthomodular lattices are Kalmbach (1982) and Ptak *et al.* (1990).

## 2. THE ORTHOMODULAR LATTICES $T(E, \varphi)$

Denote by $K$ any field, *char* $(K)$ its characteristic and $|K|$ its cardinality. Let $E$ be a three-dimensional vector-space and $\varphi : E \times E \to K$ a regular, symmetric bilinear form and $Q : E \to K$ the corresponding quadratic form. Denote by $\delta(Q)$

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the discriminant of $Q$, defined, up to a square of $K$, by the determinant of the matrix of $Q$ in any basis of $E$.

Denote by $L(E, \varphi)$ the modular lattice of all the subspaces of $E$, equipped with the polarity $M \mapsto M^{\perp}$ where $M^{\perp} = \{u \in E \mid \forall v \in M\ ,\ \varphi(u, v) = 0\}$. The elements of $L(E, \varphi)$ are $\{0\}$ , $E$ and for all $u \in E$, $u \neq 0$, $Ku$ and $(Ku)^{\perp}$. For the projective structure of $L(E, \varphi)$, the atom $Ku$ is a point and the coatom $(Ku)^{\perp}$ is a projective line.

The form $\varphi$ is regular but it can provide isotropic vectors, that is vectors $u \neq 0$ such that $Q(u) = 0$. In particular, when $K$ is finite or algebraically closed there are isotropic vectors. The isotropic subspaces of $E$, that is, the subspaces $M$ such that $M \cap M^{\perp} = \{0\}$, are the $Ku$ and the $(Ku)^{\perp}$ with $u$ an isotropic vector. Following Baer (1952, p. 106), a subspace $M$ is called an *N-subspace* if all nonzero vectors of $M$ are isotropic.

The following results can be found in Carrega *et al.* (2004). If $T(E, \varphi)$ denotes the set of all $M \in L(E, \varphi)$ such that both $M$ and $M^{\perp}$ are not $N$-subspaces then $(T(E, \varphi), \subset, \perp)$ is an orthomodular lattice (OML). In the lattice $T(E, \varphi)$, $Ku \vee_T Kv = E$ means that the two atoms $Ku$ and $Kv$ are orthogonal to a same isotropic atom in $L(E, \varphi)$.

Let $T'(E, \varphi)$ be the set of all nonisotropic $M \in L(E, \varphi)$. If char $(K) = 2$ and if $\{\omega \in E, \omega \textit{ isotropic}\} \cup \{0\}$ is a two-dimensional subspace of $E$, denoted by ${a_0}^{\perp}$, then $T'(E, \varphi)$ is the horizontal sum of $T(E, \varphi)$ and the four-element Boolean algebra $\{\{0\}, E, a_0, {a_0}^{\perp}\}$. In the other cases $T'(E, \varphi) = T(E, \varphi)$.

These OMLs were studied with Richard Greechie and René Mayet. They are interesting since they allow us to provide infinitely many minimal OMLs useful for the covering of the equational classes [MOn].

## 3. AUTOMORPHISM GROUPS AND SIMILAR FORMS

Now we want to determine the automorphism group $Aut(T(E, \varphi))$. The group $Aut(T'(E, \varphi))$ is the same or is obtained by a product with a two-element group.

We denote by $Aut_{\perp}(L(E, \varphi))$ the automorphism group of the lattice $L(E, \varphi)$ equipped with the polarity $M \mapsto M^{\perp}$; thus $g \in Aut_{\perp}(L(E, \varphi))$ means $g$ is an automorphism of the lattice $L(E, \varphi)$ and, for any $M \in L(E, \varphi)$, $g(M^{\perp}) = (g(M))^{\perp}$.

Denote by $\Phi(E)$ the set of all the regular, symmetric, bilinear forms on $E$. Two forms $\varphi$ and $\varphi'$ in $\Phi(E)$ are called *similar* if the lattices with polarity $(L(E, \varphi), \perp)$ and $(L(E, \varphi'), \perp)$ are isomorphic. This implies that the groups $Aut_{\perp}(L(E, \varphi))$ and $Aut_{\perp}(L(E, \varphi'))$ are isomorphic and the groups $Aut(T(E, \varphi))$ and $Aut(T(E, \varphi'))$ are isomorphic.

The relation "$\varphi$ similar to $\varphi'$" is an equivalent relation on $\Phi(E)$. For the study, up to isomorphism, of the groups $Aut_{\perp}(L(E, \varphi))$ and $Aut(T(E, \varphi))$ it is possible to change the form $\varphi$ for a simpler similar form.The following results can be found in Carrega *et al.* (2004).

- If $K$ is finite, there exists only one class of similar forms: More precisely every form $\varphi$ in $\Phi(E)$ is similar to the canonical form defined in some basis of $E$ by $\varphi'(u, u') = xx' + yy' + zz'$.
- If $K = \mathbb{R}$, there exist only two classes of similar forms represented by the forms expressed in some basis of $E$ by $xx' + yy' + zz'$ and $xx' + yy' - zz'$.
- If $K = \mathbb{Q}$, there exists only one class of similar forms having isotropic vectors, represented by the form expressed in some basis of $E$ by $xx' + yy' - zz'$.

## 4. ABOUT THE ISOTROPIC ATOMS IN $L(E, \varphi)$

Let $u \in E, u \neq 0, (Ku)^\perp$ is a projective line in $L(E, \varphi)$. If $(Ku)^\perp = a_0{}^\perp$ (defined in Section 2), all the atoms of $(Ku)^\perp$ are isotropic. In the other cases the number of isotropic atoms in $(Ku)^\perp$ is given by the following proposition. Recall that $Q$ is the quadratic form associated to $\varphi$ and $\delta(Q)$ the discriminant of $Q$.

**Proposition 1.** *The number of isotropic atoms in a projective line $(Ku)^\perp$, different from $(a_0)^\perp$, is given by the following.*

1. *If $Q(u) = 0$, then $Ku$ is the only isotropic atom in $(Ku)^\perp$.*
2. *If $\dfrac{-Q(u)}{\delta(Q)}$ is a nonzero square in $K$, $(Ku)^\perp$ has two isotropic atoms if* $\operatorname{char}(K) \neq 2$ *and one if* $\operatorname{char}(K) = 2$.
3. *If $\dfrac{-Q(u)}{\delta(Q)}$ is not a square in $K$, $(Ku)^\perp$ has no isotropic atoms.*

**Proof:** For the proof we need a formula which generalizes the Euclidean case. Let $B = (e_1, e_2, e_3)$ be an orthogonal basis of $E$. If $u = xe_1 + ye_2 + ze_3$ and $u' = x'e_1 + y'e_2 + z'e_3$, we have $\varphi(u, u') = axx' + byy' + czz'$ with $a = Q(e_1), b = Q(e_2), c = Q(e_3)$. As in Euclidean geometry, one can define a wedge product on $E$ $(u, u') \mapsto u \times u'$ by setting $u \times u' = bc(yz' - zy')e_1 + ac(xz' - zx')e_2 + ab(xy' - yx')e_3$. One verifies that $\varphi(u, u \times u') = \varphi(u', u \times u') = 0$, then $u \times u'$ is orthogonal to $u$ and $u'$. It is easy to verify the formula

$$\varphi^2(u, u') + \frac{Q(u \times u')}{\delta(Q)} = Q(u)Q(u')(\text{ here } \delta(Q) = abc).$$

Let $(Ku)^\perp$ be a projective line different from $a_0{}^\perp$; let $v$ and $w$ be two noncollinear vectors in $(Ku)^\perp$ with $v$ nonisotropic. The mapping $\lambda \mapsto K(\lambda v + w)$ is a bijection from $K$ onto the set of the atoms of $(Ku)^{\perp}$ different from $Kv$. We have:

$$K(\lambda v + w) \text{ isotropic } \Longleftrightarrow Q(\lambda v + w) = 0 \Longleftrightarrow \lambda^2 Q(v) + 2\lambda\varphi(v, w) + Q(w) = 0.$$

We have a second degree equation with $\Delta' = \varphi^2(v, w) - Q(v)Q(w)$. From the previous formula $\Delta' = -\frac{Q(v\times w)}{\delta(Q)}$. As $v \times w$ is collinear to $u$, we have, up to a square, $\Delta' = -\frac{Q(u)}{\delta(Q)}$.

1. If $\Delta' = 0$, then the equation has one root, hence there is only one isotropic atom in $(Ku)^{\perp}$ and this isotropic atom is $Ku$.
2. If $\Delta'$ is a nonzero square, then the equation has two roots, but if $char(K) = 2$ the two roots coincide. Hence there are two isotropic atoms in $(Ku)^{\perp}$ if $\mathrm{char}(K) \neq 2$ and only one if $\mathrm{char}(K) = 2$.
3. If $\Delta'$ is not a square, then the equation has none roots, hence there are none isotropic atoms in $(Ku)^{\perp}$. □

## 5. GROUPS LINKED TO THE QUADRATIC SPACE $(E, \varphi)$

A mapping $f : E \to E$ is called *semi-linear* if there exists $\sigma \in Aut(K)$ such that, for all $u, v \in E$ and $\lambda \in K$, $f(u + v) = f(u) + f(v)$ and $f(\lambda u) = \sigma(\lambda) f(u)$. The mapping $f$ is called *$\sigma$-linear*.

A $\sigma$-linear mapping $f : E \to E$ is called *semi-orthogonal* if for all $u$ and $v$ in $E$ $\varphi(f(u), f(v)) = \sigma(\varphi(u, v))$. The mapping $f$ is called *$\sigma$-orthogonal*. When $\sigma = id_K$, $f$ is called *orthogonal*.

Let $k \in K$, $k \neq 0$, a $\sigma$-linear mapping $f : E \to E$ is called *a similarity of coefficient k* if, for all $u$ and $v \in E$, $\varphi(f(u), f(v)) = k\sigma(\varphi(u, v))$. For instance, an homothetic transformation of ratio $h$ is a similarity of coefficient $k = h^2$.

Orthogonal, semi-orthogonal, homothetic transformations and similarity are bijections onto $E$ and we can consider the following groups:

- $O(E, \varphi)$ , the group of all orthogonal mappings.
- $O_s(E, \varphi)$, the group of all semi-orthogonal mappings.
- $\mathrm{Sim}(E, \varphi)$, the group of all similarities transformations.
- $\mathrm{Hom}(E)$, the group of all homothetic transformations.

We have $O(E, \varphi) \subset O_s(E, \varphi) \subset \mathrm{Sim}(E, \varphi)$ and $\mathrm{Hom}(E) \subset \mathrm{Sim}(E, \varphi)$.

## 6. ISOMORPHISMS

**Proposition 2.** *If $K \neq \mathbf{F}_3$ (the three-element field) then* $\mathrm{Aut}(T(E, \varphi))$ *is isomorphic to* $\mathrm{Aut}_{\perp}(L(E, \varphi))$.

**Proof:** An automorphism $g : T(E, \varphi) \to T(E, \varphi)$ can be extended to a unique automorphism $\bar{g} : L(E, \varphi) \to L(E, \varphi)$.

For defining $\bar{g}(K\omega)$ for an isotropic atom $K\omega$, we consider the set A of all the nonisotropic atoms in the isotropic line $(K\omega)^{\perp}$. For $a$ and $b$ in A, $a \neq b$, we have $a \vee_T b = E$ where $a \vee_T b$ means the join in the lattice $T(E, \varphi)$. As $g$ is an automorphism of $T(E, \varphi)$, we have $g(a) \vee_T g(b) = E$. This implies that the atoms $g(a)$ and $g(b)$ belong to a same isotropic line $(K\omega')^{\perp}$; then the atoms $g(a)$, for $a \in E$, belong pairwise to a same isotropic line.

If char$(K) = 2$, then, by Proposition 1, each atom $g(a)$ belongs to only one isotropic line, then all the atoms $g(a)$, for $a \in E$, belong to the same isotropic line $(K\omega')^{\perp}$. We define $\bar{g}(K\omega)$ by $\bar{g}(K\omega) = K\omega'$.

If char$(K) \neq 2$, then, by Proposition 1, each atom $g(a)$ belongs to two isotropic lines and these atoms are pairwise in a same isotropic line. Then $K \neq \mathbf{F}_3$ implies that all the atoms $g(a)$, for $a \in E$, belong to the same isotropic line $(K\omega')^{\perp}$. We define $\bar{g}(K\omega)$ by $\bar{g}(K\omega) = K\omega'$. For an isotropic line $(K\omega)^{\perp}$, we define $\bar{g}((K\omega)^{\perp})$ by $\bar{g}((K\omega)^{\perp}) = (\bar{g}(K\omega))^{\perp}$.

In the case where the atom $a_0$ exists, necessarily we have to set $\bar{g}(a_0) = a_0$, since $a_0$ is orthogonal to each isotropic atom and the set of all isotropic atoms is invariant under $\bar{g}$. We define $\bar{g}((a_0)^{\perp})$ by $\bar{g}((a_0)^{\perp}) = (a_0)^{\perp}$. This definition of $\bar{g}$ is necessary in order to preserve orthogonality and that implies the unicity of $\bar{g}$. Now, it si easy to verify that the extension $\bar{g}$ of $g$ is an element of $Aut_{\perp}(L(E, \varphi))$ and that the mapping $g \mapsto \bar{g}$ is a group isomorphism from $Aut(T(E, \varphi))$ to $Aut_{\perp}(L(E, \varphi))$. □

**Proposition 3.** *The group $Aut_{\perp}(L(E, \varphi))$ is isomorphic to the quotient group $Sim(E, \varphi)/Hom(E)$.*

**Proof:** We consider $\psi : \mathrm{Sim}(E, \varphi) \to \mathrm{Aut}_{\perp}(L(E, \varphi))$ defined by $\psi(f) = \bar{f}$ where $\bar{f}(M) = \{f(u) | u \in M\}$.

For proving that $\psi$ is onto, let $g \in \mathrm{Aut}_{\perp}(L(E, \varphi))$. Then

- there exists $f$ $\sigma$-linear such that $g = \bar{f}$ and the other mappings $f'$ such that $g = \bar{f}'$ are given by $f' = df$ with $d \in K^*$ (Baer, 1952, p. 44),
- there exists $k \in K$ such that for all $u, v$ in $E$ $\varphi(f(u), f(v)) = k\sigma(\varphi(u, v))$. The proof can be found in Varadarajan (1968) in a more general case, Theorem 3.1, p. 35.

If $\psi(f) = id_{L(E,\varphi)}$, by the previous Baer result, we have $f = d\ id_E$ and $f \in \mathrm{Hom}(E)$ . □

***Remark.*** When $k$ is a square, for all $g$, it is possible to obtain a similar result with the group $O_s(E, \varphi)$ instead of the group $\mathrm{Sim}(E, \varphi)$. Conditions for that will be given in the following proposition.

**Proposition 4.** *A link between* $\mathrm{Aut}_{\perp}(L(E,\varphi))$ *and* $O_{\mathrm{s}}(E,\varphi)$ *is the following:*

1. *If $K$ is algebraically closed, or if every element of $K$ is a square,*
2. *If $K$ is finite,*
3. *If $K = \mathbb{R}$,*
4. *If $K = \mathbb{Q}$ and if the form $\varphi$ has isotropic vectors*

*there exists a form $\varphi'$ similar to $\varphi$ such that* $\mathrm{Aut}_{\perp}(L(E,\varphi))$ *is isomorphic to the quotient group* $O_{\mathrm{s}}(E,\varphi')/\{id_E, -id_E\}$.

**Proof:** As in Proposition 3, we consider, for a form $\varphi'$ similar to $\varphi$, the mapping $\psi : O_{\mathrm{s}}(E,\varphi') \to \mathrm{Aut}_{\perp}(L(E,\varphi'))$ defined by $\psi(f) = \overline{f}$ where $\overline{f}(M) = \{f(u) \mid u \in M\}$.

For proving that $\psi$ is onto, let $g \in \mathrm{Aut}_{\perp}(L(E,\varphi'))$. Then there exists a $\sigma-$linear mapping $f$ such that $g = \overline{f}$ and there exists $k \in K$ such that for all $u, v$ in $E$, $\varphi'(f(u), f(v)) = k\sigma(\varphi'(u,v))$ (1). When $v = u$ the formula (1) becomes:

For all $u$ in $E$ $Q'(f(u)) = k\sigma(Q'(u))$ (2) where $Q'$ is the quadratic form associated to $\varphi'$. It follows from these relations that

"$u$ isotropic" $\Longleftrightarrow$ "$f(u)$ isotropic" and

"$v$ isotropic and orthogonal to $u$"$\Longleftrightarrow$ "$f(v)$ isotropic and orthogonal to $f(u)$," where the words isotropic and orthogonal refer to the form $\varphi'$. Then, for $u \neq 0$, the vectors $u$ and $f(u)$ are of the same type: isotropic, nonisotropic and orthogonal to isotropic vectors, nonisotropic and nonorthogonal to isotropic vectors. By Proposition 1, it follows that for each nonisotropic vector $u$ in $E$ the elements of $K$, $\frac{-Q'(u)}{\delta(Q')}$, and $\frac{-Q'f(u))}{\delta(Q')}$, are both squares or are both nonsquares. In each case given in Proposition 4, we will prove that we can choose $\varphi'$ such that the element $k$ in the relations (1) and (2) is a square.

In case (1), if every element of $K$ is a square, the result is obvious, we choose $\varphi' = \varphi$.

In cases (2), (3), (4), from Section 3, we can choose a form $\varphi'$ similar to $\varphi$ defined in some basis of $E$ by $\varphi'(u,u') = xx' + yy' + zz'$ or $\varphi'(u,u') = xx' + yy' - zz'$. For these two forms $\delta(Q') = \pm 1$ and the relation (2) can be written $\frac{-Q'(f(u))}{\delta(Q')} = k\ \sigma(\frac{-Q'(u)}{\delta(Q')})$ (3). Then, for every nonisotropic vector $u$, $k$ is a quotient of two elements of $K$ which are both squares or nonsquares. In cases (2) and (3) this implies that $k$ is a square, because in a finite field or in the field $\mathbb{R}$ the quotient of two nonsquares is a square. In case (4) with $K = \mathbb{Q}$, we have choosen the form expressed in some basis $B = (e_1, e_2, e_3)$ by $\varphi'(u,u') = xx' + yy' - zz'$. With this form we have $Q'(e_1) = 1$, $\delta(Q') = -1$ and $\frac{-Q'(e_1)}{\delta(Q')} = 1$ is a square, then relation (3), written with $u = e_1$, implies that $k$ is a square.

Then we set $k = d^{-2}$ for $d \in K$ and we have, by using relation (1), for all $u$, $v$ in $E$, $\varphi'(df(u), df(v)) = \sigma(\varphi'(u,v))$. Then $df \in O_{\mathrm{s}}(E,\varphi')$ and $g = \overline{f} = \overline{df}$.

If $f \in \mathrm{Ker}(\psi)$, then $\overline{f} = \overline{id_E}$ and by a result of Baer (1952, p. 44), we have $f = h\ id_E$ with $h \in K^*$. As $f$ is linear and belongs to $O_{\mathrm{s}}(E,\varphi)$, we have

$f \in O(E, \varphi)$. Then, for all $u$ and $v$ in $E$ we have $\varphi'(hu, hv) = \varphi'(u, v)$ and this implies $h^2 = 1$, $h = \pm 1$ and $\mathrm{Ker}(\psi) = \{\pm id_E\}$. □

***Remarks.***

(1) If all elements of $K$ are squares, then $\varphi' = \varphi$.
(2) If $K = \mathbb{R}$ or $K = \mathbb{Q}$, then $O_s(E, \varphi') = O(E, \varphi')$.
(3) If $\mathrm{char}(K) = 2$, then $O_s(E, \varphi')/\{id_E, -id_E\} = O_s(E, \varphi')$.

From Propositions 2 and 3 we obtain :

**Theorem 1.** *If $K \neq \mathbf{F}_3$, $Aut(T(E, \varphi))$ is isomorphic to* $\mathrm{Sim}(E, \varphi)/\mathrm{Hom}(E)$.

From Propositions 2 and 4 we obtain :

**Theorem 2.** *A link between $Aut(T(E, \varphi))$ and $O_s(E, \varphi)$ is the following:*

1. *If $K$ is algebraically closed, or if every element of $K$ is a square,*
2. *If $K$ is finite and $K \neq \mathbf{F}_3$,*
3. *If $K = \mathbb{R}$,*
4. *If $K = \mathbb{Q}$ and if the form $\varphi$ has isotropic vectors*

*there exists a form $\varphi'$ similar to $\varphi$ such that $Aut(T(E, \varphi))$ is isomorphic to the quotient group $O_s(E, \varphi')/\{id_E, -id_E\}$.*

## 7. CARDINALITY OF *AUT*$(T(E, \varphi))$ WHEN $K$ IS FINITE

When $K$ is a finite field, $|K| = q = p^n$ with $p$ a prime number and $n \geq 1$. Up to isomorphism, the orthomodular lattice $T(E, \varphi)$ is independent of the regular and symmetric bilinear form $\varphi$; it depends only on the field $K$, for that we denote $T(E, \varphi) = T_q$.

We can take for $\varphi$ a canonical form given in a basis $B = (e_1, e_2, e_3)$ by

$$\varphi(xe_1 + ye_2 + ze_3, x'e_1 + y'e_2 + z'e_3) = xx' + yy' + zz'$$

and

$$Q(xe_1 + ye_2 + ze_3) = x^2 + y^2 + z^2.$$

**Theorem 3.** *For $q = p^n$, if $q \neq 3$, then* $|\mathrm{Aut}(T_q)| = nq(q^2 - 1)$.

**Proof:**

- From the homomorphism $f \longmapsto \sigma_f$ from $O_s(E, \varphi)$ to $Aut(K)$, we obtain $|O_s(E, \varphi)| = |O(E, \varphi)||Aut(K)|$.

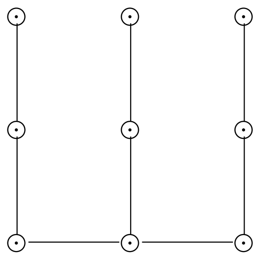
- If $p \neq 2$, from Theorem 2, $|Aut(T_q)| = \frac{1}{2}|O_s(E, \varphi)|$ and from the previous result $|Aut(T_q)| = \frac{1}{2}|O(E, \varphi)||Aut(K)|$.
  The cardinality $|O(E, \varphi)|$ is the number of ordered orthogonal bases $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ of $E$ such that $Q(\varepsilon_i) = 1$ for $i = 1, 2, 3$. The number of blocks of $T_q$ corresponding to $(K\varepsilon_1, K\varepsilon_2, K\varepsilon_3)$ is $\frac{q(q^2-1)}{24}$, then $|O_s(E, \varphi)| = 2^3 \times 3! \frac{q(q^2-1)}{24} = 2q(q^2 - 1)$. On the other hand, we have $|Aut(K)| = n$, hence $|\mathrm{Aut}(T_q)| = nq(q^2 - 1)$.
- If $p = 2$, from Theorem 2,

$$|\mathrm{Aut}(T_q)| = |O(E, \varphi)||\mathrm{Aut}(K)|$$

  with $|O(E, \varphi)| = 3! \frac{q(q^2-1)}{6} = q(q^2 - 1)$ and $|\mathrm{Aut}(K)| = n$ and therefore we obtain the same result $|\mathrm{Aut}(T_q)| = nq(q^2 - 1)$. □

***Remarks.***

(1) The Greechie diagram of $T_3$ is the following



and for this OML, Theorem 3 does not work, but directly we can find $|\mathrm{Aut}(T_3)| = 2^3 \times 3! = 48$.

(2) We know that $T_4 = G_{32}$ and that $Aut(G_{32})$ is isomorphic to the symmetric group $S_5$, hence $|\mathrm{Aut}(T_4)| = 120$. The same result is given by the formula $nq(q^2 - 1)$ with $q = 4$ and $n = 2$.

The following theorem explains why the OMLs $T_q$ are symmetric. But before the theorem we have to recall some things about the OMLs $T_q$.

- When $q = p^n$ is odd ($p \neq 2$), there are in $T_q$ two kinds of atoms according to the number of blocks passing through, and that implies two kinds of blocks.
- When $q = 2^n$, there is one kind of atom and one kind of block.

**Theorem 4.** *If $B$ and $B'$ are two blocks of $T_q$ of the same type, there exists $g \in \mathrm{Aut}(T_q)$ such that $g(B) = B'$.*

*Roughly speaking this theorem says that blocks of the same type play the same role.*

**Proof:** As the two blocks $B$ and $B'$ are of the same type, it is possible to find vectors, representing the atoms of these blocks, such that $B = (K\varepsilon_1, K\varepsilon_2, K\varepsilon_3)$, $B' = (K\varepsilon'_1, K\varepsilon'_2, K\varepsilon'_3)$ and, for $i = 1, 2, 3$, $Q(\varepsilon_i) = Q(\varepsilon'_i)$. Then the linear mapping $f$ sending the orthogonal basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ to the orthogonal basis $(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3)$ is in $O(E, \varphi)$. Define $g$ by $g = \bar{f}$, that is, $g(M) = \{f(u)/u \in M\}$, we have $g$ in $\mathrm{Aut}(T_q)$ and we have, for $i = 1, 2, 3$, $g(K\varepsilon_i) = K\varepsilon'_i$. □

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